THE MODELLING AND ANALYSIS OF FRACTIONAL-ORDER CONTROL SYSTEMS IN DISCRETE DOMAIN

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Abstract

This paper deals with fractional-order controlled systems and fractional-order controllers in the discrete domain. The mathematical description by the fractional difference equations and properties of these systems are presented. A practical example for modelling the fractional-order control loop is shown and obtained results are discussed in conclusion.

Key words: discrete fractional calculus, digital controller, discrete fractional-order system.

1. INTRODUCTION

Fractional calculus was used for modelling of physical systems, but we can find only few works dealing with the application of this mathematical tool in control theory (e.g. [1, 2, 5, 8, 7, 9]). These works used continuous mathematical models of fractional order. The fractional-order systems have a unlimited memory, being integer-order systems cases in which the memory is limited.

It is necessary to realise that the case of discrete fractional-order systems is very important for their description to have a finite difference equation. Such an equation can be obtained by numerical approximation and by the inverse Z-transform of a discrete transfer function.

The aim of this paper is to show, how by using the fractional calculus, we can obtain a more general fractional-order models of the controlled objects and general structure for the classical PID controller in the discrete domain.

2. DISCRETE FRACTIONAL CALCULUS

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695.

Fractional calculus is a generalisation of integration and differentiation to non-integer order fundamental operator ${}_aD_t^\alpha$, where a and t are the limits of the operation. The continuous integro-differential operator is defined as

$$
{}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha} & \Re(\alpha) > 0, \\ 1 & \Re(\alpha) = 0, \\ \int_a^t (d\tau)^{-\alpha} & \Re(\alpha) < 0. \end{cases}
$$

The two definitions used for the general fractional differintegral are the Grünwald-Letnikov (GL) definition and the Riemann-Liouville (RL) definition [6]. The GL is given here

$$ {}_aD_t^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} f(t - jh), \tag{1} $$

where $[x]$ means the integer part of x. The RL definition is given as

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{2} $$

for $(n - 1 < \alpha < n)$ and where $\Gamma(.)$ is the well known Euler's *Gamma* function.

In general, the approximation for fractional operator of order α can be expressed by generating function $\omega(\xi^{-1})$, where ξ^{-1} is the shift operator. This generating function and its expansion determine both the form of the approximation and the coefficients [4].

For discrete approximation of the time derivative, we can use the generating function corresponding to the Z-transform of backward difference rule,

$$ \omega(z^{-1}) = \frac{1 - z^{-1}}{T}, \tag{3} $$

and performing the power series expansion (PSE) of $(1 - z^{-1})^{\pm\alpha}$, we obtain the Z version of the GL formula by using the short memory principle [9], for the discrete equivalent of the fractional-order integro-differential operator $\omega(z^{-1})^{\pm\alpha}$,

$$ (\omega(z^{-1}))^{\pm\alpha} = T^{\mp\alpha} \sum_{j=0}^{[\frac{L}{T}]} (-1)^j \binom{\pm\alpha}{j} z^{[\frac{L}{T}]-j}, \tag{4} $$

where T is the sample period, L is the memory length, $[x]$ is the integer part of x and $(-1)^j \binom{\pm\alpha}{j}$ are a binomial coefficients $c_j^{(\alpha)}$, $(j = 0, 1, \ldots)$. For its calculation we can use the following expression:

$$ c_0^{(\alpha)} = 1, \qquad c_j^{(\alpha)} = \left(1 - \frac{1 + (\pm\alpha)}{j}\right) c_{j-1}^{(\alpha)}. \tag{5} $$

Another possibility for the discrete approximation is the use of the trapezoidal (Tustin) rule as a generating function for PSE. For the discrete equivalent of the fractional-order integro-differential operator, we can write a general formula [5]

$$ (\omega(z^{-1}))^{\pm\alpha} = \left(\frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}\right)^{\pm\alpha} \tag{6} $$

for obtaining the coefficients and the form of the approximation.

A detailed review of the approximation methods (Calson's, Chareff's, CFE, Matsuda's, ...) for continuous and discrete fractional-order models was done in work [10].

3. FRACTIONAL-ORDER CONTROL CIRCUIT

We will be studying the control system shown in Fig.1, where $G_c(z)$ is the controller transfer function, $G_s(z)$ is the controlled system transfer function, $W(z)$ is an input, $E(z)$ is an error, $U(z)$ is the output from controller and $Y(z)$ is the output from system [3].

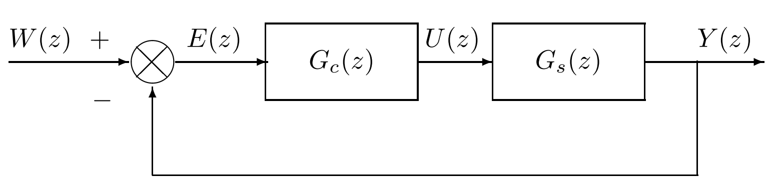

Figure 1: *Feed - back control loop*

3.1 Fractional-order controlled system

The fractional-order controlled system will be represented with a fractional model with the fractional differential equation given by the following expression (${}_0D_t^{\mu} \equiv D_t^{\mu}$):

$$a_n\, D_t^{\beta_n} y(t) + \ldots + a_1\, D_t^{\beta_1} y(t) + a_0\, D_t^{\beta_0} y(t) = b_n\, D_t^{\alpha_n} u(t) + \ldots + b_1\, D_t^{\alpha_1} u(t) + b_0\, D_t^{\alpha_0} u(t), \tag{7}$$

where β_k, α_k $(k = 0, 1, 2, \ldots)$ are generally real numbers, $\beta_n > \ldots > \beta_1 > \beta_0$, $\alpha_m > \ldots > \alpha_1 > \alpha_0$ and a_k, b_k $(k = 0, 1, \ldots)$ are arbitrary constants.

For obtaining a discrete model of the fractional-order system (7), we have to use discrete approximations of the fractional-order integro-differential operators and then we obtain a general expression for the discrete transfer function of the controlled system [10]

$$G_s(z) = \frac{b_m(\omega(z^{-1}))^{\alpha_m} + \ldots + b_1(\omega(z^{-1}))^{\alpha_1} + b_0(\omega(z^{-1}))^{\alpha_0}}{a_n(\omega(z^{-1}))^{\beta_n} + \ldots + a_1(\omega(z^{-1}))^{\beta_1} + a_0(\omega(z^{-1}))^{\beta_0}}, \tag{8}$$

where $(\omega(z^{-1}))$ denotes the discrete operator, expressed as a function of the complex variable z or the shift operator z^{-1}.

For discrete time step k, according to relation (4) and the inverse Z-transform of the difference equation (7), we can write the fractional-order difference equation in the form

$$\begin{aligned} &a_n\, T^{-\beta_n} \sum_{j=0}^{k} c_j^{(\beta_n)} y_{k-j} + \ldots + a_1\, T^{-\beta_1} \sum_{j=0}^{k} c_j^{(\beta_1)} y_{k-j} + a_0\, T^{-\beta_0} \sum_{j=0}^{k} c_j^{(\beta_0)} y_{k-j} = \\ &b_m\, T^{-\alpha_m} \sum_{j=0}^{k} c_j^{(\alpha_m)} u_{k-j} + \ldots + b_1\, T^{-\alpha_1} \sum_{j=0}^{k} c_j^{(\alpha_1)} u_{k-j} + b_0\, T^{-\alpha_0} \sum_{j=0}^{k} c_j^{(\alpha_0)} u_{k-j}. \end{aligned} \tag{9}$$

From relation (9) the following form of the difference equation can be obtained

$$y_k = \frac{\sum_{i=1}^{m}(b_i T^{-\alpha_i} \sum_{j=0}^{k} c_j^{(\alpha_i)} u_{k-j}) - \sum_{i=1}^{n}(a_i T^{-\beta_i} \sum_{j=1}^{k} c_j^{(\beta_i)} y_{k-j})}{\sum_{i=0}^{n} a_i T^{-\beta_i} c_0^{(\beta_i)}}, \tag{10}$$

for $k = 2, 3, \ldots$, where $y_0 = 0$ and $y_1 = 0$.

3.2 Fractional order controller

The fractional $PI^{\lambda}D^{\delta}$ controller will be represented by discrete transfer function given in the following expression:

$$G_c(z) = \frac{U(z)}{E(z)} = K + \frac{T_i}{(\omega(z^{-1}))^{\lambda}} + T_d(\omega(z^{-1}))^{\delta}, \tag{11}$$

where λ and δ are arbitrary real numbers $(\lambda, \delta \geq 0)$, K is the proportional constant, T_i is the integration constant and T_d is the derivative constant.

Taking $\lambda = 1$ and $\delta = 1$, we obtain a classical PID controller. If $\lambda = 0$ and/or $T_i = 0$, we obtain a PD^{δ} controller, etc. All these types of controllers are particular cases of the $PI^{\lambda}D^{\delta}$ controller, which is more flexible and gives an opportunity to better adjust the dynamical properties of the fractional-order control system.

Similar to methods for controlled system, they can be used for obtaining the difference equation of the fractional-order controller (11).

4. ILLUSTRATIVE EXAMPLE

We give in this section an example of modelling the stable dynamical system by using fractional calculus in discrete domain. The fractional-order control system consists of the real controlled system with the coefficients:

$$a_2 = 0.8,\ a_1 = 0.5,\ a_0 = 1.0,\ \beta_2 = 2.2,\ \beta_1 = 0.9,\ \beta_0 = 0,\ b_0 = 1.0,\ \alpha_0 = 0 \tag{12}$$

and the fractional-order PD^{δ} controller, designed on the stability measure $S_t = 2.0$ and damping measure $\xi = 0.4$, with the coefficients:

$$K = 50.0,\ T_d = 5.326,\ \delta = 1.286. \tag{13}$$

The fractional-order differential equation of the closed control loop has the form (${}_0D_t^{\mu} \equiv D_t^{\mu}$):

$$a_2\, D_t^{\beta_2} y(t) + a_1\, D_t^{\beta_1} y(t) + T_d\, D_t^{\delta} y(t) + (a_0 + K) y(t) = K\ w(t) + T_d\, D_t^{\delta} w(t). \tag{14}$$

The resulting differential equation (14) can be rewritten to the fractional difference equation of the closed control loop in general form via inverse Z-transform of the relation (4):

$$y_k = \frac{K w_k + T_d T^{-\delta} \sum\limits_{j=0}^{k} c_j^{(\delta)} w_{k-j} - a_2 T^{-\beta_2} \sum\limits_{j=1}^{k} c_j^{(\beta_2)} y_{k-j} - a_1 T^{-\beta_1} \sum\limits_{j=1}^{k} c_j^{(\beta_1)} y_{k-j} - T_d T^{-\delta} \sum\limits_{j=1}^{k} c_j^{(\delta)} y_{k-j}}{a_2 T^{-\beta_2} c_0^{(\beta_2)} + a_1 T^{-\beta_1} c_0^{(\beta_1)} + T_d T^{-\delta} c_0^{(\delta)} + (a_0 + K)}, \tag{15}$$

for $(k = 1, 2, \ldots)$, where $y_0 = 0,\ w_0 = 0,\ w_1 = 0$ and $w_k = 1$, for $(k = 2, 3, \ldots)$. The binomial coefficients were calculated according to relation (5).

5. CONCLUSION

The above methods make it possible to model and analyse (simulate) fractional-order control systems in the discrete domain and also to realise digital fractional-order controllers. We have shown the survey of approximation methods and Z-transform method as good approximations of the fractional-order operator ${}_aD_t^{\alpha}$. These methods for discretisation of fractional calculus are suitable for realisation and implementation of fractional-order controllers, because this controllers are more robust than classical one.

Stability investigation of discrete fractional-order control system and practical simulation will be studied in further work.

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